Implementation of NURBS Curve Derivatives in Engineering Practice

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ABSTRACT

Derivatives are a very important tool of computations in an engineering practice on the graphics structures. B-Spline functions are defined recursively, so the direct computation is very difficult. In this article new direct proof of the formula used for simpler direct computation is shown. The paper also presents own method of programming derivatives of NURBS curves by means of the proven formula which is implemented in German engineering software RFEM 3D.

Keywords: NURBS, derivative.

1. INTRODUCTION

NURBS (Non-Uniform Rational B-Splines) are used in graphic software to design free-form objects and also in various engineering software for designing buildings, bridges, etc. Derivatives are necessary in these applications for physical computations such as stress analysis.

The first part of this paper introducing a new direct proof of the formula used for computing derivatives of B-Spline functions. The gist of this proof is that we don't need any other characteristics of B-Spline function as in the case of the proof in [PT92]. Therefore our proof is more straightforward and clear. Second part of paper presents our method of programming derivatives of the NURBS. We deal with verifying of the continuity with respect to a special case and all other issues which are necessary for succesfull implemenation of the derivatives.

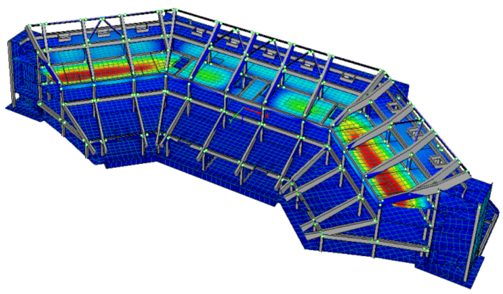

Figure 1: Stress Analysis in RFEM3D – Ice Hockey Stadium

2. RELATED WORK

Our work with NURBS object has started by studying and implementation of curves, general surfaces (see [Pro06a]) and basic types of NURBS solids – cone, cylinder, sphere, cube (see [Pro06b]) to RFEM 3D CAD system. As mentioned, the derivatives are the key for computing stress analysis, therefore we started analyze them. First conception is published in [Pro05].

The theoretical background for this paper is mostly described in three sources. First one is a book by Peigl and Tiller [PT92] about NURBS objects. This book contains the algorithms of computing points on NURBS curve and reveal the proof of derivatives. Second source is a Floater's paper [Flo92b] about derivatives of Bézier curves. In third one [Flo92a] two ways of the first derivative of rational B-Spline curve in terms of its control points and weights are described.

3. B-SPLINE AND NURBS

This section briefly outlines the theory about B-Spline and NURBS used in the rest of paper. B-Spline (NURBS) are defined with B-Spline functions recursively. Let $t = (t_0, t_1, \dots, t_n)$ be a knot vector (non-decreasing sequence of real numbers). Then B-Spline function is defined by:

$$N_i^0(t) = \begin{cases} 1 & \text{for} \quad t \in \langle t_i, t_i+1) \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

$$N_i^k(t) = \frac{t - t_i}{t_{i+k} - t_i} N_i^{k-1}(t) + \frac{t_{i+k+1} - t}{t_{i+k+1} - t_{i+1}} N_{i+1}^{k-1}(t),$$

where $0 \le i \le n-k-1, 1 \le k \le n-1, \frac{0}{0} := 0$.

We added the control points $P_0, P_1, \dots, P_m$ $(P_i \in \mathbf{R}^d)$ and the knot vector $\mathbf{t} = (t_0, t_1, \dots t_{m+n+1})$. Then B-Spline curve is defined:

$$C(t) = \sum_{i=0}^{m} P_i N_i^n(t) \tag{2}$$

Let $w_0, \ldots, w_m$ be a vector of weights, which are attached to every control points. Weights (non-negative numbers) determine the influence of its control point. So, NURBS curve of degree n is defined by:

$$C(t) = \frac{\sum_{i=0}^{m} w_i P_i N_i^n(t)}{\sum_{i=0}^{m} w_i N_i^n(t)} \tag{3}$$

where $t \in \langle t_n, t_{m+1} \rangle$.

4. PROOF OF THE DERIVATIVE FORMULA

Let us show the new proof of the formula for computations of NURBS curve derivatives.

Theorem 1. *Let $C(t)$ be the B-Spline curve from previous section. Then its first derivative is evaluated by:*

$$C(t)' = \sum_{i=0}^{m} N_i^n(t)' P_i, \tag{4}$$

where

$$N_i^n(t)' = \frac{n}{t_{i+n} - t_i} N_i^{n-1}(t) - \frac{n}{t_{i+n+1} - t_{i+1}} N_{i+1}^{n-1}(t).$$

Proof: The proof will be done by complete induction with respect to n.
1. $n = 0$
Obviously, the derivative is equal to zero in all cases. The theorem holds for $n = 0$.
2. Let us suppose, that the formula holds for $k = 0, 1, \ldots, n$. The idea of proof is in comparing different expressions of $N_i^{n+1}(t)'$. The first is done by computation of the expression:

$$N_i^{n+1}(t)' =$$

$$\left(\frac{t - t_i}{t_{i+n+1} - t_i} N_i^n(t) + \frac{t_{i+n+2} - t}{t_{i+n+2} - t_{i+1}} N_{i+1}^n(t) \right)'.$$

According to the induction suppose, we know the expressions of degree n: $N_i^n(t)'$ and $N_{i+1}^n(t)'$ and we substitute them. The second expression is from Theorem 1 for degree $n+1$. We get these two summands:

$$\begin{aligned}
N_i^{n+1}(t)' &= \frac{1}{t_{i+n+1} - t_i} N_i^n(t) - \frac{1}{t_{i+n+2} - t_{i+1}} N_{i+1}^n(t) \\
&+ \frac{n}{t_{i+n+1} - t_i} \left(\frac{t - t_i}{t_{i+n} - t_i} N_i^{n-1}(t) \right) \\
&- \frac{n}{t_{i+n+2} - t_{i+1}} \left(\frac{t_{i+n+2} - t}{t_{i+n+2} - t_{i+2}} N_{i+2}^{n-1}(t) \right) \\
&+ \frac{n}{t_{i+n+1} - t_i} \left(\frac{t_{i+n+1} - t}{t_{i+n+1} - t_{i+1}} N_{i+1}^{n-1}(t) \right) \\
&- \frac{n}{t_{i+n+2} - t_{i+1}} \left(\frac{t - t_{i+1}}{t_{i+n+1} - t_{i+1}} N_{i+1}^{n-1}(t) \right)
\end{aligned}$$

and

$$\begin{aligned}
N_i^{n+1}(t)' &= \frac{1}{t_{i+n+1} - t_i} N_i^n(t) - \frac{1}{t_{i+n+2} - t_{i+1}} N_{i+1}^n(t) \\
&+ \frac{t - t_i}{t_{i+n+1} - t_i} \left(\frac{n}{t_{i+n} - t_i} N_i^{n-1}(t) \right) \\
&- \frac{t_{i+n+2} - t}{t_{i+n+2} - t_{i+1}} \left(\frac{n}{t_{i+n+2} - t_{i+2}} N_{i+2}^{n-1}(t) \right) \\
&- \frac{t - t_i}{t_{i+n+1} - t_i} \left(\frac{n}{t_{i+n+1} - t_{i+1}} N_{i+1}^{n-1}(t) \right) \\
&+ \frac{t_{i+n+2} - t}{t_{i+n+2} - t_{i+1}} \left(\frac{n}{t_{i+n+1} - t_{i+1}} N_{i+1}^{n-1}(t) \right)
\end{aligned}$$

First three parts are identical. The equality is not evident for the last two expressions. Multinomial nN_{i+1}^{n-1} is exclude and we can make a special step – in the numerator we add and subtract $t_{i+1} t_{i+n+1}$. Then we got the desired equality. This finishes the proof and implies:

Corollary 1. *Let $C(t)$ be a NURBS curve defined in previous section. Then its derivative is of the form:*

$$\begin{aligned}
C(t)' &= \left(\frac{\sum_{i=0}^{m} w_i P_i N_i^n(t)}{\sum_{i=0}^{m} w_i N_i^n(t)} \right)' = \\
&= \frac{\sum_{i=0}^{m} w_i P_i N_i^n(t)' \sum_{i=0}^{m} w_i N_i^n(t)}{\left(\sum_{i=0}^{m} w_i N_i^n(t)\right)^2} \\
&- \frac{\sum_{i=0}^{m} w_i P_i N_i^n(t) \sum_{i=0}^{m} w_i N_i^n(t)'}{\left(\sum_{i=0}^{m} w_i N_i^n(t)\right)^2}.
\end{aligned} \tag{5}$$

It's proven, that derivative formula holds and can be used in engineering computations without any limitations.

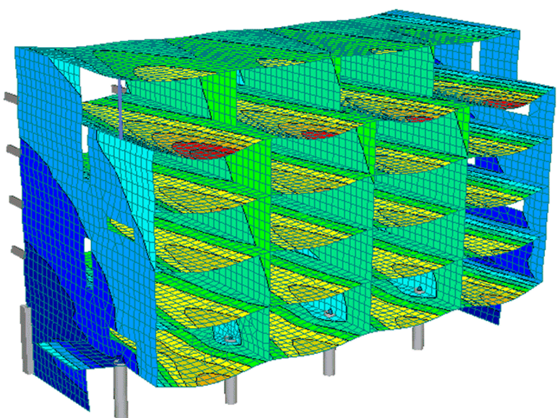

Figure 2: Analysis of Stress – Building

5. IMPLEMENTATION OF DERIVATIVES

The algorithm has two parts. The first one is the verification of continuity of $C(t)$ for particular input value of parameter t, the second one contains the computations.

Verification of Continuity

Let $C(t)$ be a NURBS curve of degree n defined in theoretical part of article. First, we have to define the multiplicity of one knot in the knot vector. The value $\bar{t}$ has multiplicity s, when knot vector $\mathbf{t}$ contains value $\bar{t}$ s-times. Then, $C(t)$ is C_{n-s} continuous at the point $C(\bar{t})$ if the derivative of $(n-s)$-th order in $C(t)$ exists.
Let us now summarize the verification of curve continuity.

1) We find the interval of the knot vector with $\bar{t} \in \langle t_i, t_{i+1})$.

2) We find the multiplicity of $\bar{t}$.

3) We compute the continuity in the point $C(\bar{t})$.

4) We treat the special cases.

Special Case

Some curves have a points, which continuity is C_0 but the tangent and derivatives exist here. Generally:
Let have B-Spline curve $C(t)$ with arbitrary control point $P_i = C(\bar{t})$. $\bar{t}$ has multiplicity equal to degree of curve, then the continuity in point P_i is C_0. But if the control points P_{i-1}, P_i, P_{i+1} lie in the line then the derivative in the middle one exists.

Example of this case should be a circle. Figure 3 shows such circle as NURBS. For the parameter $t = 0.25$ the continuity in the point P_2 equals zero. However points P_1, P_2, P_3 lie in the line then the derivative in P_2 exists – it is line P_1P_3.

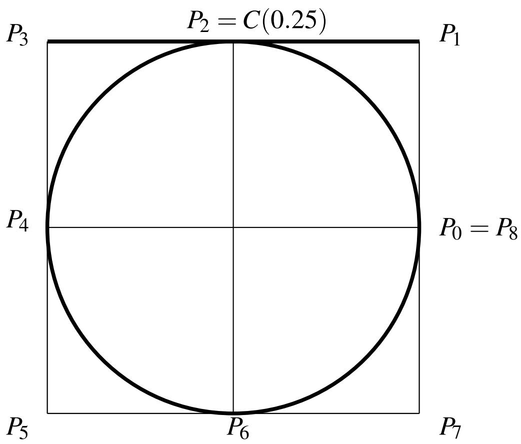

Figure 3: NURBS circle, degree = 2, knot vector $(0,0,0,0.25,0.25,0.5,0.5,0.75,0.75,1,1,1)$

Computation

The computation has two parts: initialization and computational algorithm.

a) Inicialization

- Inicialize basic functions N_i^n.

```
if t in <t_i, t_i+1)
  N[0][i] = 1;
  N[0][j] = 0; (j != i)
```

- Make projectivization of the control points (i.e. connecting the points with their weights).

```
for (i=0; i<array_length; i++)
  Point4d[i][0] =
  weights[i]*ControlPoint[i];
```

b) Computation

It is necessary to divide the computation of formula (5) in Corollary 1 into several parts.

Multinominal $N_i^j(t), j = 1,..,n$, where n is the degree of the curve, is computed by deBoor algorithm (see [Bo76], [PT92]).
Multinominal $N_i^n(t)'$ is computed by use of the proven formula in previous section means of N_i^{n-1} (deBoor algorithm for degree $n-1$).

Let us remark that the zero denominator is treated separately by defining $0/0 = 0$ which is typical for NURBS.

6. DERIVATIVES ON SURFACE

B-Spline (NURBS) surface is tensor product of two B-Spline (NURBS) curves with its parameters. Computing of partial derivatives is analogous to the curves. We obtain these derivatives by computing derivatives of basis functions N_i^p and N_j^q.

$$\frac{\partial^{k+s}}{\partial^k u \partial^s v} S(u,v) = \sum_{i=0}^{n} \sum_{j=0}^{m} N_{i,p}^{(k)} N_{j,q}^{(s)} P_{ij} \tag{6}$$

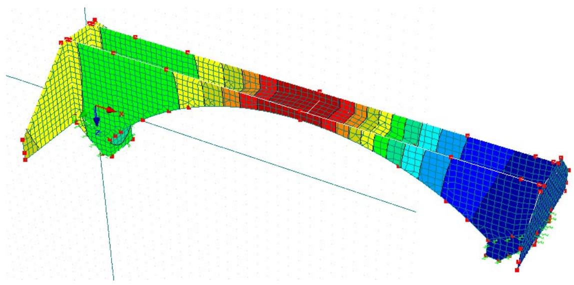

Figure 4: Analysis of Stress – Bridge

7. CONCLUSION

The derivatives of NURBS curves are in literature discussed only briefly and theoretically, therefore whole theory and its implementation is described in this article. Their importance in technical practice is enormous – physical calculations, building industry and so on.

The discussion in this article has given an overview of derivatives of B-Spline (NURBS). We demonstrated a new proof of known formula for computation of derivatives, which is more clear and straightforward. We also reveal the possibility of programming with respect to the verification of the continuity and special case.

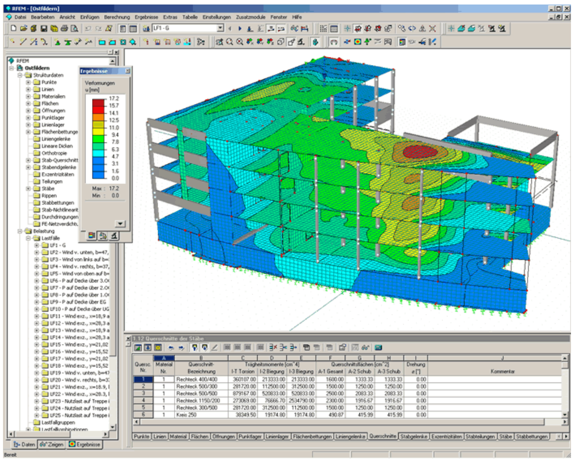

Figure 5: RFEM 3D

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